Exercise 29

Solve the boundary-value problem, if possible.

$$y'' = y', \quad y(0) = 1, \quad y(1) = 2$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2e^{rx}$$

Plug these formulas into the ODE.

$$r^2e^{rx} = re^{rx}$$

Divide both sides by e^{rx}.

$$r^2 = r$$

Solve for r.

$$r^2 - r = 0$$

$$r(r-1) = 0$$

$$r = \{0, 1\}$$

Two solutions to the ODE are $e^0 = 1$ and e^x. By the principle of superposition, then,

$$y(x) = C_1 + C_2e^x.$$

Apply the boundary conditions to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 1$$

$$y(1) = C_1 + C_2e = 2$$

Solving this system of equations yields $C_1 = (e-2)/(e-1)$ and $C_2 = 1/(e-1)$. Therefore, the solution to the boundary value problem is

$$\begin{aligned} y(x) &= \frac{e-2}{e-1} + \frac{1}{e-1}e^x \\ &= \frac{e-2+e^x}{e-1}. \end{aligned}$$

Below is a graph of $y(x)$ versus x.